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Numerical Solution of the 2-Dimensional Euler-Bernoulli Beam Equation by using Finite Difference Method

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Abstract

The Euler-Bernoulli beam theory is a simplification of the linear theory of elasticity which provides a means of calculating the load-carrying and deflection characteristics of beams. The beam equation, $EI\frac{\partial^4 u}{\partial x^4} = -p\frac{\partial^2 u}{\partial t^2}$ describes the relationship between the beam's deflection, $u(x, t)$ and the applied load, $p(x, t)$. This equation is widely used in engineering practices. When designing bridges and buildings, the engineers are interested in determining deflections because the beam may be in direct contact with a brittle material such as glass. Although analytical solutions to Partial Differential Equations (PDEs) are exact, they may not be easy to solve and in most cases, the solutions are in closed form. This makes numerical solutions ideal for such calculations. From the existing literature, the discussion on the beam equation is not exhaustive. It is therefore the aim of the study to investigate the numerical solution of the equation of structural analysis of a beam that incorporates the longitudinal movement. This study has managed to solve numerically the 4th order 2-dimensional beam equation, $u_{tt}(x, y, t) + \alpha^2[u_{xxxx}(x, y, t) + u_{yyyy}(x, y, t)] = f(x, y, t)$ using the finite difference method subject to special boundary and initial conditions. The study has checked the accuracy of the numerical scheme by analyzing its stability and convergence. The results of this study indicate that the new algorithm has small computational work, faster convergence speed and high precision.

Keywords

Euler-Bernoulli Beam Equation, Centered Time Centered Space, Finite Difference Representation, Courant Fredrichs Lewy Condition, Computational Domain.

1 Introduction

The Euler-Bernoulli beam equation is given by:

$$EI\frac{d^4 u(x)}{dx^4} = p(x) \tag{1.1}$$

where E is the Elastic modulus, I is the Second moment of area and $p(x)$ is the distributed load. The transverse motion of a uniform Euler-Bernoulli beam equation, (1.1) as modelled by Zhang, H.L., [22], in his Journal publication, gave the Numerical Solution of Euler-Bernoulli Beam Equation by using Barycentric Lagrange Interpolation Collocation Method. It has been discussed and analyzed extensively in beam structures but ignores the longitudinal movements. Therefore, there is need to address the beam displacement which incorporates the longitudinal movement as below:

$$u_{tt}(x, y, t) + \alpha^2[u_{xxxx}(x, y, t) + u_{yyyy}(x, y, t)] = f(x, y, t) \tag{1.2}$$

where $EI = \alpha$ is a constant and $f(x, y, t)$ is the forcing term.

For this reason, the study has managed to solve the 4th order 2-dimensional beam equation (1.2) subject to the following special boundary conditions:

$$\begin{aligned} u(0, y, t) &= 0, 0 \leq y \leq 1, t > 0, \\ u(1, y, t) &= 0, 0 \leq y \leq 1, t > 0, \\ u(x, 0, t) &= 0, 0 \leq x \leq 1, t > 0, \\ u(x, 1, t) &= 0, 0 \leq x \leq 1, t > 0 \end{aligned} \tag{1.3}$$

with initial conditions:

$$\begin{aligned} u(x, y, 0) &= (\sin\pi x)(\sin\pi y) = g_1(x, y) \\ u_t(x, y, 0) &= g_2(x, y) = 0 \end{aligned} \tag{1.4}$$

2 Method of Solution

The concept of the finite difference algorithms has been applied. The explicit (Centered Time Centered Space) scheme for solving the general beam equation subject to special boundary conditions and with consistent initial conditions has been used. Error analysis for the scheme developed has been done using Taylor's series expansion to determine the order of accuracy. The stability and consistency of the numerical methods has also been analyzed to test their efficacy. Finally, the graphical outputs of both the 1-dimensional analytical and the numerical solutions have been compared with the 2-dimensional solution.

3 Numerical Scheme and Stability Analysis

3.1 Computational Domain

The computational domain, Ω is assumed to be rectangular with x in $x_{min} \leq x \leq x_{max}$, y in $y_{min} \leq y \leq y_{max}$ and t in $0 \leq t \leq T$. The discrete approximation of the wave field $u(x_m, y_n, t^l) = u^l_{m,n}$. Here, $x_m = m\Delta x$, $y_n = n\Delta y$ and $t^l = l\Delta t$, for all $m = 1, 2, 3...M - 1$, $n = 1, 2, 3...N - 1$ and $l = 0, 1, 2...L - 1$. It is important to note that m, n, l are discrete and finite. Also, $\Delta x = h$ is the grid size in x-direction (transverse), $\Delta y = k$ is the grid size in the y-direction (longitudinal) and $\Delta t = s$ represents the increment in time, t. Taking a uniform grid both in space and time, then $h = k = s$.

3.2 Discretization of equation (1.2)

The equation, $u_{tt}(x, y, t) + \alpha^2[u_{xxxx}(x, y, t) + u_{yyyy}(x, y, t)] = f(x, y, t)$ is a linear hyperbolic PDE. It is discretized using the Finite Difference approximations as below:

Since x and y are spatial co-ordinates and t is time, the finite difference approximations to the partial derivatives w.r.t x is given by:

$$u_{xxxx} = \frac{1}{4h^4}[u^l_{m+3,n} - 2u^l_{m+2,n} - u^l_{m+1,n} + 4u^l_{m,n} - u^l_{m-1,n} - 2u^l_{m-2,n} + u^l_{m-3,n}] \tag{3.1}$$

In the same way, the finite difference approximations w.r.t y is:

$$u_{yyyy} = \frac{1}{4k^4}[u^l_{m,n+3} - 2u^l_{m,n+2} - u^l_{m,n+1} + 4u^l_{m,n} - u^l_{m,n-1} - 2u^l_{m,n-2} + u^l_{m,n-3}] \tag{3.2}$$

And, $u_{tt}(x, y, t)$ is approximated by:

$$u_{tt}(x, y, t) = \frac{1}{s^2}[u^{l+1}_{m,n} - 2u^l_{m,n} + u^{l-1}_{m,n}] \tag{3.3}$$

The function, $f(x, y, t)$ is the forcing term which merely changes the amplitude of the forced vibration of the system with two degrees of freedom. It is an independent term and is approximated by $f^l_{m,n}$, [16, 19].

3.3 Explicit scheme (Centered Time Centered Space)

Here, $u_{tt}(x, y, t)$, $u_{xxxx}(x, y, t)$ and $u_{yyyy}(x, y, t)$ are replaced by their central difference approximations respectively to get;

$$\frac{1}{s^2}[u^{l+1}_{m,n} - 2u^l_{m,n} + u^{l-1}_{m,n}] + \alpha^2\{\frac{1}{4h^4}[u^l_{m+3,n} - 2u^l_{m+2,n} - u^l_{m+1,n} + 4u^l_{m,n} - u^l_{m-1,n} - 2u^l_{m-2,n} + u^l_{m-3,n}]$$

$$+ \frac{1}{4k^4}[u^l_{m,n+3} - 2u^l_{m,n+2} - u^l_{m,n+1} + 4u^l_{m,n} - u^l_{m,n-1} - 2u^l_{m,n-2} + u^l_{m,n-3}]\} + O(h^2, k^2, s^2) = f^l_{m,n} \tag{3.4}$$

It is second order accurate both in space and time.

For a discrete grid of equal mesh ratio, $h = k$. Taking $\frac{\alpha^2}{h^4} = \frac{\alpha^2}{k^4} = r$, gives:

$$\frac{1}{s^2}u^{l+1}_{m,n} + \frac{1}{4}r[u^l_{m+3,n} - 2u^l_{m+2,n} - u^l_{m+1,n} + u^l_{m,n+3} - 2u^l_{m,n+2} - u^l_{m,n+1}] + (2r - \frac{2}{s^2})u^l_{m,n}$$

$$-f^l_{m,n} = -\frac{1}{s^2}u^{l-1}_{m,n} - \frac{1}{4}r[-u^l_{m-1,n} - 2u^l_{m-2,n} + u^l_{m-3,n} - u^l_{m,n-1} - 2u^l_{m,n-2} + u^l_{m,n-3}] \tag{3.5}$$

for all $m = 1, 2, 3..., M - 1$, $n = 1, 2, 3..., N - 1$ and $l = 0, 1, 2...L - 1$.

Therefore, equation (3.5) is the Explicit Scheme (Centered Time Centered Space) for the equation (1.2).

3.4 Von Neumann Stability analysis

In order to analyze the stability of the solution, we let $u^l_{m,n} = \xi^l e^{ik(x_m+y_n)}$, where ξ is an amplitude factor and k is some constant.

Substituting into equation (3.5), ignoring the forcing term, $f^l_{m,n}$ and re-arranging, we get;

$$\frac{1}{s^2}\xi^{(l+1)}e^{ikx_m}e^{iky_n} + \frac{1}{4}r[\xi^l e^{ikx_{m+3}}e^{iky_n} - 2\xi^l e^{ikx_{m+2}}e^{iky_n} - \xi^l e^{ikx_{m+1}}e^{iky_n} + \xi^l e^{ikx_m}e^{iky_{n+3}}$$

$$-2\xi^l e^{ikx_m} e^{iky_{n+2}} - \xi^l e^{ikx_m} e^{iky_{n+1}}] + (2r - \frac{2}{s^2})\xi^l e^{ikx_m} e^{iky_n} + \frac{1}{s^2}\xi^{(l-1)} e^{ikx_m} e^{iky_n}$$
$$+\frac{1}{4}r[-\xi^l e^{ikx_{m-1}} e^{iky_n} - 2\xi^l e^{ikx_{m-2}} e^{iky_n} + \xi^l e^{ikx_{m-3}} e^{iky_n} - \xi^l e^{ikx_m} e^{iky_{n-1}} - 2\xi^l e^{ikx_m} e^{iky_{n-2}}$$
$$+\xi^l e^{ikx_m} e^{iky_{n-3}}] = 0 \qquad (3.6)$$

Dividing through by $\xi^l e^{ikx_m} e^{iky_n}$ and simplifying to gives:

$$\xi^2 + \frac{1}{2}rs^2\xi[\cos 3k\Delta x - 2\cos 2k\Delta x - \cos k\Delta x + \cos 3k\Delta y - 2\cos 2k\Delta y - \cos k\Delta y] +$$
$$2rs^2\xi - 2\xi + 1 = 0 \qquad (3.7)$$

which is a quadratic equation in ξ.

From the quadratic equation (3.7); $a = 1$, $b = \frac{1}{2}rs^2[\cos 3k\Delta x - 2\cos 2k\Delta x - \cos k\Delta x + \cos 3k\Delta y - 2\cos 2k\Delta y - \cos k\Delta y] + 2rs^2 - 2$ and $c = 1$

Solving for ξ, we get:

$$\xi = \frac{1}{2}\{-\omega \pm \sqrt{\omega^2 - 4}\} \qquad (3.8)$$

where:

$\omega = \frac{1}{2}rs^2[\cos 3k\Delta x - 2\cos 2k\Delta x - \cos k\Delta x + \cos 3k\Delta y - 2\cos 2k\Delta y - \cos k\Delta y] + 2rs^2 - 2$

Solving for dissipation, $|\xi|^2$ gives:

$$|\xi|^2 = \frac{1}{2}(\omega^2 \pm \omega\sqrt{\omega^2 - 4}) - 1 \qquad (3.9)$$

But, the maximum value of $\cos\phi = 1$, hence $\omega = -2$.

This implies that:

$$|\xi|^2 = 1 \qquad (3.10)$$

The Von Neumann stability criteria, $|\xi|^2 \leq 1$ is satisfied, hence, the CTCS scheme is unconditionally stable.

4 Results and Discussion

4.1 Results

Using the MATHEMATICA program, we run equation (3.5) for the Explicit Scheme (Centered Time Centered Space). This creates a 3-D square mesh grid with $t_{max} = 1$, $x_{max} = 1$ and $y_{max} = 1$.

From the special boundary conditions, equation (1.3) and initial boundary conditions, equation (1.4), we find $u^0_{m,n}$ i.e at time level 0

$$u^0_{m,n} = u(x_m, y_n, 0) = \sin(\pi x_m)\sin(\pi y_n) \qquad (4.1)$$

This generates the values of the solution when $t = 0$ as shown in figure (4.1) for the Centered Time Centered Space.

We then generate the values for time level 1, ie $u^1_{m,n}$. In the same way, the subsequent time levels are obtained.

The trial input values produce the results of displacement which incorporates longitudinal movements at varying time levels for the finite difference schemes as shown in figure (4.2).

The Numerical and Analytical solutions of the 4th order 1 - dimensional beam equation in (1.1) are as follows in figures (4.3) and (4.4) respectively, [22]

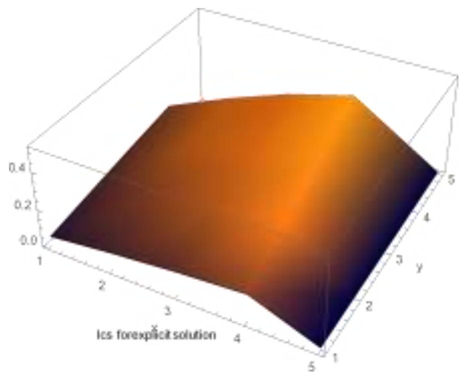

Figure 4.1: Initial Conditions for Explicit Solution (CTCS)

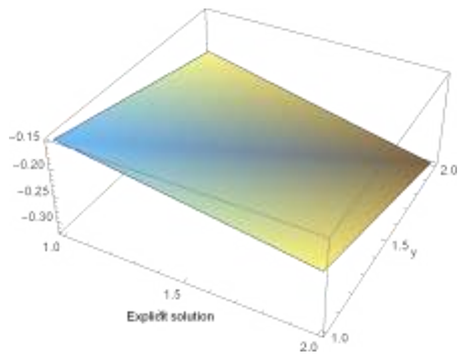

Figure 4.2: Explicit Solution (CTCS)

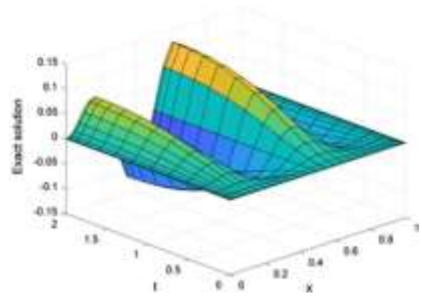

Figure 4.3: Analytical solution of 1 - dimensional Euler-Bernoulli Beam Equation

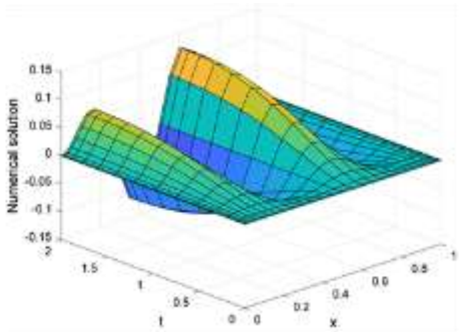

Figure 4.4: Numerical solution of 1 - dimensional Euler-Bernoulli Beam Equation

Table 1: Comparison of Results from the graphical outputs

x	t	Analytical, 1-D	Numerical, 1-D	Explicit (CTCS), 2-D
0.0990	0.0251	-0.0019	-0.0018	-0.0096
0.7939	1.0000	-0.0315	-0.0315	-0.0430
0.1654	1.9010	0.0651	0.0651	0.0538

4.2 Discussions

The arbitrary nature of the displacement amplitude implies that under the right conditions, very large displacements can be experienced by the beam, i.e., the beam can resonate. The results of displacement of the graphical solutions in figures, (4.2), (4.3) and (4.4) show a common trend but differ with small margins. This could be due to the introduction of the longitudinal component. When a load is applied to a beam structure, it deflects and the magnitude of displacement increases with more load until it corresponds to the natural frequency of the beam. It then decreases due to the suppression from the longitudinal movement.

5 Conclusions

The study has managed to develop a numerical scheme for the 4th order 2-dimensional Euler-Bernoulli Beam equation i.e Centered Time Centered Space.

Von Neumann stability analysis reveals that the scheme is stable on the Courant-Fredrichs-Lewy (CFL) condition.

The results of this study compared with other methods show that this method has high accuracy and faster convergence. However, the rate of convergence of the algorithm depends so much on the truncation errors introduced when approximating the partial derivatives.

It is also worth noting that the smaller the mesh ratios or sizes, the better the results since this makes the grids finer thus improving the approximations within the boundaries. This however takes more computational time as evidenced in the figure (4.2).

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