International Advance Journal of Engineering Research (IAJER)
Volume 3, Issue 1 (January- 2020), PP 07-11
ISSN: 2360-819X
www.iajer.com

Research Paper **Open Access**

Solution of Heat and Wave Equations using Mahgoub Adomian Decomposition Method

Mohammad Almousa, Imad Al-zuhairi, Yousef Al-Qudah and Hamza Qoqazeh
(Department of Mathematics, Faculty of Arts and Science, Amman Arab University, Amman, Jordan)
**Corresponding Author: Mohammad Almousa, Imad Al-zuhairi*

ABSTRACT : *In this paper, Mahgoub Adomian decomposition method (MADM), to handle the wave and heat equations, is introduced. The efficiency of the present method will be shown by applying the procedure on four examples.*
Keywords – *Mahgoub Adomian decomposition method, Mahjoub transform, Partial Differential Equations, Heat equation, Wave equation.*

I. INTRODUCTION

An $n\text{-}th$ order PDE for $u(x_1, x_2, x_3, \dots, x_n)$, $n \geq 2$ is a relation of the form

$$F(\vec{x}, u, Du, D^2u, \dots, D^nu) = 0, \tag{1}$$

where F is neither linear or nonlinear function.

The focus will be on two main types of PDEs as well, namely wave and heat equations. Now, we are going to define these equations as

$$\frac{\partial^2 u}{\partial t^2} = c^2\left(\frac{\partial^2 u}{\partial {x_1}^2} + \frac{\partial^2 u}{\partial {x_2}^2} + \dots + \frac{\partial^2 u}{\partial {x_n}^2}\right) \tag{2}$$

$$\frac{\partial u}{\partial t^2} = c^2\left(\frac{\partial^2 u}{\partial {x_1}^2} + \frac{\partial^2 u}{\partial {x_2}^2} + \dots + \frac{\partial^2 u}{\partial {x_n}^2}\right) \tag{3}$$

respectively. The wave and heat equations have many possible applications in mathematics, physics and engineering $(see\ [1-6])$.

In 2016, Mahgoub [7] define the function A for $t \geq 0$

$$A = \left\{u(t): \exists M, k_1, k_2 > 0, |u(t)| < Me^{\frac{|t|}{k_j}}\right\}, \tag{4}$$

where M, k_1, k_2 are fixed and M is a finite.

The operator $M\{u(t)\}$ may be expanded as

$$M\{u(t)\} = H(v) = v\int_0^\infty u(t)e^{-vt}\,dt, \quad k_1 \leq v \leq k_2. \tag{5}$$

Many linear and nonlinear problems like ODEs, PDEs, Integral equations and integro- differential equations were solved using Mahgoub transform and ADM$(see\ [8-13])$.

In present paper, we have application MADM for solving wave and heat equations.

II. APPLICATION OF THE METHOD

The general inhomogeneous form of PDE is

$$Lu(x,t) + Ru(x,t) + Nu(x,t) = h(x,t) \tag{6}$$

with

$$u(x,0) = f(x) \qquad u_t(x,0) = g(x)$$

where $L = \frac{\partial^2}{\partial t^2}$, with inverse $L_u^{-1} = \int_0^t\int_0^t (.)dtdt$, R is linear operator and N is nonlinear operator.

Now, we rewrite (6) as

$$Lu(x,t) = h(x,t) - Ru(x,t) - Nu(x,t) \tag{7}$$

Take Mahgoub transform to both sides, we have

$$M\{Lu(x,t)\} = M\{h(x,t) - Ru(x,t) - Nu(x,t)\} \tag{8}$$

Linearity of $M\{.\}$ yields

$$M\{Lu(x,t)\} = M\{h(x,t)\} - M\{Ru(x,t)\} - M\{Nu(x,t)\} \tag{9}$$

Solving for (9), we get

$$v^2 M\{u(x,t)\} - u(x,0)v^2 - v u_t(x,0) = M\{h(x,t)\} - M\{Ru(x,t)\} - M\{Nu(x,t)\} \quad (10)$$

By substituting $u(x,0)$ and $u_t(x,0)$ into (10), we obtain

$$v^2 M\{u(x,t)\} - f(x)v^2 - v g(x) = M\{h(x,t)\} - M\{Ru(x,t)\} - M\{Nu(x,t)\} \quad (11)$$

This leads to

$$M\{u(x,t)\} = f(x) + \frac{1}{v} g(x) + \frac{1}{v^2} M\{h(x,t)\} - \frac{1}{v^2} M\{Ru(x,t)\} - \frac{1}{v^2} M\{Nu(x,t)\} \quad (12)$$

Replacing $u(x,t) = \sum_{n=0}^{\infty} u_n(x,t)$ and $Nu(x,t) = \sum_{n=0}^{\infty} A_n$ in (12), we have

$$M\{\sum_{n=0}^{\infty} u_n(x,t))\} = f(x) + \frac{1}{v} g(x) + \frac{1}{v^2} M\{h(x,t)\} - \frac{1}{v^2} M\{R \sum_{n=0}^{\infty} u_n(x,t)\} - \frac{1}{v^2} M\{\sum_{n=0}^{\infty} A_n\} \quad (13)$$

where

$$A_n = \frac{1}{n!} \frac{d^n}{d\lambda^n} [\sum_{i=0}^{\infty} \lambda^n N u_i]_{\lambda=0}, \quad m = 0,1,\ldots \quad (14)$$

Therefore, we have

$$M\{u_0(x,t)\} = f(x) + \frac{1}{v} g(x) + \frac{1}{v^2} M\{h(x,t)\}$$

$$M\{u_{n+1}(x,t)\} = -\frac{1}{v^2} (M\{R \sum_{n=0}^{\infty} u_n(x,t) + M\{\sum_{n=0}^{\infty} A_n\}\}), \; n \geq 0 \quad (15)$$

Applying M^{-1} to both sides, yields

$$u_0(x,t) = M^{-1}\{f(x)\} + t g(x) + M^{-1}\left\{\frac{1}{v^2} M\{h(x,t)\}\right\}$$

$$u_{n+1}(x,t) = -M^{-1}\left\{\frac{1}{v^2} (M\{R \sum_{n=0}^{\infty} u_n(x,t) + M\{\sum_{n=0}^{\infty} A_n\}\})\right\}, \; n \geq 0 \quad (16)$$

III. Applications and Results

Example 1: Consider the wave equation [5]

$$u_{tt} = \frac{1}{2} x^2 u_{xx}, \quad (17)$$

Subject to $u(x,0) = x$, $u_x(x,0) = x^2$.
We use $M\{.\}$ to both sides, yields

$$M\{u_{tt}\} = M\left\{\frac{1}{2} x^2 u_{xx}\right\}, \quad (18)$$

which gives

$$v^2 M\{u(x,t)\} - u(x,0)v - v u_t(x,0) = M\left\{\frac{1}{2} x^2 u_{xx}\right\} \quad (19)$$

By substituting $u(x,0) = x$, $u_x(x,0) = x^2$ into (19), we obtain

$$v^2 M\{u(x,t)\} - x v^2 - v x^2 = M\left\{\frac{1}{2} x^2 u_{xx}\right\} \quad (20)$$

Replacing $u(x,t) = \sum_{n=0}^{\infty} u_n(x,t)$ in (20), we have

$$M\{\sum_{n=0}^{\infty} u_n(x,t)\} = x + \frac{1}{v} x^2 + \frac{1}{v^2} M\left\{\frac{1}{2} x^2 u_{n_{xx}}\right\}, \; n \geq 0 \quad (21)$$

This leads to

$$M\{u_0(x,t)\} = x + \frac{1}{v} x^2$$

$$M\{u_{n+1}(x,t)\} = \frac{1}{v^2} M\left\{\frac{1}{2} x^2 u_{n_{xx}}\right\}, \; n \geq 0 \quad (22)$$

Applying M^{-1} to both sides, yields

$$u_0(x,t) = x + t x^2$$

$$u_{n+1}(x,t) = M^{-1}\left\{\frac{1}{v^2} M\left\{\frac{1}{2} x^2 u_{n_{xx}}\right\}\right\}, \; n \geq 0 \quad (23)$$

The solutions of (23) are given by

$$u_0(x,t) = x + t x^2$$

$$u_1(x,t) = M^{-1}\left\{\frac{1}{v^2} M\left\{\frac{1}{2} x^2 u_{0_{xx}}\right\}\right\} = M^{-1}\left\{\frac{1}{v^2} M\left\{\frac{1}{2} x^2 (2t)\right\}\right\} = x^2 M^{-1}\left\{\frac{1}{v^3}\right\} = \frac{1}{3!} t^3 x^2$$

$$u_2(x,t) = M^{-1}\left\{\frac{1}{v^2} M\left\{\frac{1}{2} x^2 u_{1_{xx}}\right\}\right\} = M^{-1}\left\{\frac{1}{v^2} M\left\{\frac{1}{2} x^2 \left(\frac{2}{3!} t^3\right)\right\}\right\} = \left(\frac{x^2}{3!}\right) M^{-1}\left\{\frac{1}{v^5}\right\} = \frac{x^2 t^5}{5!}$$

$$u_3(x,t) = M^{-1}\left\{\frac{1}{v^2} M\left\{\frac{1}{2} x^2 u_{2_{xx}}\right\}\right\} = M^{-1}\left\{\frac{1}{v^2} M\left\{\frac{1}{2} x^2 \left(\frac{2}{5!} t^5\right)\right\}\right\} = \frac{1}{7!} x^2 M^{-1}\left\{\left\{\frac{1}{v^7}\right\}\right\} = \frac{x^2 t^7}{7!}$$

Thus

$$\begin{aligned} u(x,t) &= \sum_{n=0}^{\infty} u_n(x,t) = x + t x^2 + \frac{x^2 t^3}{3!} + \frac{x^2 t^5}{5!} + \cdots \\ &= x + x^2 \left(t + \frac{t^3}{3!} + \frac{t^5}{5!} + \frac{t^7}{7!} + \cdots\right) \\ &= x + x^2 \sinh t \end{aligned} \quad (24)$$

Example 2: Consider the Heat equation [4]

$$u_t = u_{xx}\,, \tag{25}$$

Subject to $u(x,0) = ae^{kx}$, where a and k are constants.
We use $M\{.\}$ to both sides, yields

$$M\{u_t\} = M\{u_{xx}\}, \tag{26}$$

which gives

$$vM\{u(x,t)\} - vu(x,0) = M\{u_{xx}\} \tag{27}$$

By substituting $u(x,0) = ae^{kx}$ into (27), we obtain

$$vM\{u(x,t)\} - ae^{kx}v = M\{u_{xx}\} \Rightarrow M\{y(x,t)\} = ae^{kx} + \frac{1}{v}M\{u_{xx}\} \tag{28}$$

Replacing $u(x,t) = \sum_{n=0}^{\infty} u_n(x,t)$ in (28), we have

$$M\{\textstyle\sum_{n=0}^{\infty} u_n(x,t)\} = ae^{kx} + \frac{1}{v}M\{u_{n_{xx}}\} \tag{29}$$

This leads to

$$M\{u_0(x,t)\} = ae^{kx}$$

$$M\{u_{n+1}(x,t)\} = \frac{1}{v}M\{u_{n_{xx}}\},\ n \geq 0 \tag{30}$$

Applying M^{-1} to both sides, yields

$$u_0(x,t) = ae^{kx}$$

$$u_{n+1}(x,t) = M^{-1}\left\{\frac{1}{v}M\{u_{n_{xx}}\}\right\},\ n \geq 0 \tag{31}$$

The solutions of (31) are given by

$$u_0(x,t) = ae^{kx}$$

$$u_1(x,t) = M^{-1}\left\{\frac{1}{v}M\{u_{0_{xx}}\}\right\} = M^{-1}\left\{\frac{1}{v}M\{ak^2e^{kx}\}\right\} = ak^2e^{kx}M^{-1}\left\{\frac{1}{v}\right\} = ak^2te^{kx}$$

$$u_2(x,t) = M^{-1}\left\{\frac{1}{v}M\{u_{1_{xx}}\}\right\} = M^{-1}\left\{\frac{1}{v}M\{ak^4te^{kx}\}\right\} = M^{-1}\left\{ak^4e^{kx}\frac{1}{v}M\{t\}\right\}$$

$$= ak^4e^{kx}M^{-1}\left\{\frac{1}{v^2}\right\} = ak^4\frac{t^2}{2!}e^{kx}$$

$$u_3(x,t) = M^{-1}\left\{\frac{1}{v}M\{u_{2_{xx}}\}\right\} = M^{-1}\left\{\frac{1}{v}M\left\{ak^6\frac{t^2}{2!}e^{kx}\right\}\right\} = M^{-1}\left\{\frac{ak^6e^{kx}}{2!}\frac{1}{v}M\{t^2\}\right\}$$

$$= \frac{ak^6e^{kx}}{2!}M^{-1}\left\{\frac{2}{v^3}\right\} = ak^6\frac{t^3}{3!}e^{kx}$$

Thus

$$u(x,t) = \sum_{n=0}^{\infty} u_n(x,t) = ae^{kx} + ak^2te^{kx} + ak^4\frac{t^2}{2!}e^{kx} + \cdots$$

$$= ae^{kx}\left(1 + k^2t + k^4\frac{t^2}{2!} + k^6\frac{t^3}{3!} + \cdots\right)$$

$$= ae^{kx}e^{k^2t}$$

$$= ae^{kx+k^2t} \tag{32}$$

Example 3: The following nonlinear heat equation [4]

$$u_t + uu_x = u_{xx}\,, \tag{33}$$

with $u(x,0) = 2x$.
Mahgoub transform of (33) is

$$M\{u_t\} = M\{u_{xx}\} - M\{uu_x\}, \tag{34}$$

Therefore

$$vM\{u(x,t)\} - vu(x,0) = M\{u_{xx}\} - M\{uu_x\} \tag{35}$$

By substituting $u(x,0) = 2x$ into (35), we find that

$$M\{u(x,t)\} = 2x + \frac{1}{v}M\{u_{xx}\} - \frac{1}{v}M\{uu_x\} \tag{36}$$

Replacing $u(x,t) = \sum_{n=0}^{\infty} u_n(x,t)$ and $Nu(x,t) = \sum_{n=0}^{\infty} A_n$ in (36) and then applying M^{-1}, yields

$$u_0(x,t) = 2x$$

$$u_{n+1}(x,t) = M^{-1}\left\{\frac{1}{v}\left(M\{u_{n_{xx}}\} - M\{A_n\}\right)\right\},\ n \geq 0 \tag{37}$$

Therefore the solutions of (37) are

$$u_0(x,t) = 2x$$

$$u_1(x,t) = M^{-1}\left\{\frac{1}{v}\left(M\{u_{0_{xx}}\} - M\{A_0\}\right)\right\} = M^{-1}\left\{\frac{1}{v}\left(M\{0\} - M\{u_0u_x\}\right)\right\} = M^{-1}\left\{-\frac{1}{v}4x\right\}$$

$$= -4xM\left\{\frac{1}{v}\right\} = -4xt$$

$$u_2(x,t) = M^{-1}\left(\frac{1}{v}\left(M\{u_{1_{xx}}\} - M\{A_1\}\right)\right) = M^{-1}\left\{\frac{1}{v}\left(M\{0\} - M\{u_{0_x}u_1 + u_{1_x}u_0\}\right)\right\}$$

$$= M^{-1}\left\{-\frac{1}{v}M\{-16xt\}\right\} = 16xM^{-1}\left\{\frac{1}{v}M\{t\}\right\} = 16xM^{-1}\left\{\frac{1}{v^2}\right\} = 16x\frac{t^2}{2!}$$

$$u_3(x,t) = M^{-1}\left\{\frac{1}{v}\left(M\{u_{2_{xx}}\} - M\{A_2\}\right)\right\} = -96x\frac{t^3}{3!},$$

and so on. Then, the solution is

$$u(x,t) = \sum_{n=0}^{\infty} u_n(x,t) = -4xt + 16x\frac{t^2}{2!} - 96x\frac{t^3}{3!} + \cdots$$

$$= \frac{2x}{1+2t}. \qquad (38)$$

Example 4: Solve the nonlinear wave equation [5]

$$u_t + u^2{}_x = 0\,, \qquad (39)$$

Subject to $u(x,0) = -x^2$.

Take $M\{.\}$ to both sides, we have

$$M\{u_t\} = -M\{u^2{}_x\}, \qquad (40)$$

And

$$vM\{u(x,t)\} - vu(x,0) = -M\{u^2{}_x\} \qquad (41)$$

Thus

$$M\{u(x,t)\} = -x^2 - \frac{1}{v}\,M\{u^2{}_x\}) \qquad (42)$$

Using $u(x,t) = \sum_{n=0}^{\infty} u_n(x,t)$ and $Nu(x,t) = \sum_{n=0}^{\infty} A_n$ in (42) and then taking M^{-1}, yields

$$u_0(x,t) = -x^2$$

$$u_{n+1}(x,t) = -M^{-1}\left\{\frac{1}{v}\,(M\{A_n\})\right\},\ n \geq 0 \qquad (43)$$

Therefore the solutions of (43) are given

$$u_0(x,t) = -x^2$$

$$u_1(x,t) = -M^{-1}\left\{\frac{1}{v}M(A_0)\right\} = -M^{-1}\left\{\frac{1}{v}M\{u^2{}_{0_x}\}\right\} = -M^{-1}\left\{\frac{4x^2}{v}\right\}$$

$$= -4x^2M\left\{\frac{1}{v}\right\} = -4x^2t$$

$$u_2(x,t) = -M^{-1}\left\{\frac{1}{v}M\{A_1\}\right\} = -M^{-1}\left\{\frac{1}{v}M\{2u_{0_x}u_{1_x}\}\right\} = -2M^{-1}\left\{\frac{1}{v}M\{16x^2t\}\right\}$$

$$= -32x^2M^{-1}\left\{\frac{1}{v}M\{t\}\right\} = -32x^2M^{-1}\left\{\frac{1}{v^2}\right\} = -32x^2\frac{t^2}{2!} = -16x^2t^2$$

$$u_3(x,t) = -M^{-1}\left\{\frac{1}{v}M\{A_2\}\right\} = -64x^2t^3\,,$$

$$u_m(x,t) = -M^{-1}\left\{\frac{1}{v}M\{A_{m-1}\}\right\} = -4^m x^2 t^m,\ \ m = ,1,2,\dots$$

Then, the solution is

$$u(x,t) = \sum_{n=0}^{\infty} u_n(x,t) = -x^2 - 4x^2t - 16x^2t^2 + \cdots$$

$$= \frac{x^2}{1+4t}. \qquad (44)$$

IV. CONCLUSION

Mahgoub Adomian decomposition method to the wave and heat equations was introduced. The efficiency of the method has been shown by applying the procedure on some examples.

REFERENCES

[1]. J. He and X. Wu, Exp-function method for nonlinear wave equations, *Chaos, Solitons & Fractals, 30(3),* 2006, 700-708.

[2]. J. He, Application of homotopy perturbation method to nonlinear wave equations, *Chaos, Solitons & Fractals, 26(3),* 2005, 695-700.

[3]. A.-M.Wazwaz, A sine-cosine method for handlingnonlinear wave equations, *Mathematical and Computer Modelling, 40(5-6),* 2004, 499-508.

[4]. R. I. Nuruddeen and A. M. Nass, Aboodh Decomposition Method and its Application in Solving Linear and Nonlinear Heat Equations, *European Journal of Advances in Engineering and Technology, 3(7),* 2016, 34-37.

[5]. R. I. Nuruddeen and A. M Nass, Exact Solutions of Wave-Type Equations by the Aboodh Decomposition Method, *Stochastic Modelling and Applications, 21(1),* 217, 23-30.

[6]. Yun-Mei Zhao, New Exact Solutions for a Higher-Order Wave Equation of KdV Type Using the Multiple Simplest Equation Method, *Journal of Applied Mathematics, Volume 2014,* Article ID 848069, 13 pages, https://doi.org/10.1155/2014/848069.

[7]. M. Mahgoub, The New Integral Transform "Mahgoub Transform", *Advances in Theoretical and Applied Mathematics, 11(4),* 2016, 391-398.

[8]. M. Almousa, Solution of Second Order Initial- Boundary Value Problems of Partial Integro-Differential Equations by using a New Transform: Mahgoub Transform, *European Journal of Advances in Engineering and Technology, 5(10),* 2018, 802-805.

[9]. M. Almousa, A. Ismail and A. Rashid, A numerical method for solving Urysohn type integral equations, *International Journal of Mathematical Analysis, 8(13),* 2014, 1133 - 1141.

[10]. M. Almousa, A. Ismail, Equivalence between homotopy perturbation method and Adomian decomposition method in solving two dimensional linear Fredholm integral equations, *AIP Conference Proceedings, 1635 (1),* 2014, 45-51.

[11]. M. Almousa, The solutions of three dimensional Fredholm integral equations using Adomian decomposition method, *AIP Conference Proceedings, 1739(1),* 2016, https://doi.org/10.1063/1.4952533.

[12]. M. Almousa, O. N. Ogilat, A new approach for solution of linear time-delay integral equations via Z-decommission method, *Far East Journal of Mathematical Sciences, 102(11),* 2017, 2551-2561.

[13]. G. Adomian, A review of the decomposition method in applied mathematics, *Journal of Mathematical Analysis and Applications, 135(2),* 1988, 501-544.

[14]. M. Mahgoub, A. Alshikh, An Application of New Transform " Mahgoub Transform" to Partial Differential Equations, *Mathematical Theory and Modeling , 7(1),* 2017, 7-9.

[15]. P. Senthil Kumar and A. Viswanathan, Applications of Mahgoub Transform to Mechanics Electrical Circuit Problems, *International Journal of Science and Research (IJSR), 7(7),* 2018, 195-197.